Series of O.D.E

حل المعادلة التفاضلية باستخدام المتسلسلات

1. $y'' + P(x)\, y' + Q(x)\, y = f(x)$

$\boxed{1}$ O.P $\quad P(0) \neq \infty \quad , \quad Q(0) \neq \infty$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$\boxed{ex}$ $y'' + y = 0 \quad . \; x_0 = 0$

$\boxed{الحل}$

$P(x) = 0 \quad , \quad Q(x) = 1$

$P(0) = 0 \neq \infty \; , \; Q(0) = 1 \neq \infty \quad \Rightarrow (O.P)$

$$y = \sum a_n x^n$$

$$y' = \sum n a_n x^{n-1} \Rightarrow y'' = \sum n(n-1) a_n x^{n-2}$$

$$\sum n(n-1) a_n x^{n-2} + \sum a_n x^n = 0$$

→ Coeff. of x^0

$$2a_2 + a_0 = 0 \Rightarrow a_2 = \frac{-a_0}{2}$$

→ Coeff. of x^1

$$6a_3 + a_1 = 0 \rightarrow a_3 = \frac{-a_1}{6}$$

$\rightarrow$ coeff. of x^n :-

$$(n+2)(n+1)\,a_{n+2} + a_n = 0$$

$$\boxed{a_{n+2} = \frac{-a_n}{(n+2)(n+1)}}$$

$\underline{\underline{n=2}}$ $\quad a_4 = \frac{-a_2}{12} = \frac{a_0}{24}$

$\underline{\underline{n=3}}$ $\quad a_5 = \frac{-a_3}{20} = \frac{a_1}{120}$

$$y = a_0 + a_1 x + \frac{-a_0}{2} x^2 + \frac{-a_1}{6} x^3 + \frac{a_0}{24} x^4 \ldots\ldots$$

الأفكار:

① $x_0 \neq 0$ ← $t = x - x_0$

$\boxed{ex}$ $y'' + x y' + y = 0$ $\qquad x_0 = 1 \longrightarrow t = x - 1$

$$y'' + (t+1) y' + y = 0 \quad , \quad t_0 = 0 \qquad y = \sum a_n t^n$$

② شرطي المسألة تحل عادي وفي الآخر تحسب a_0, a_1.

③ كثيرة حدود ← تساوي معاملات الطرف الأيمن.

④ دوال مثلثية - لوغاريتمية - أسية ← تقوم بفكها.

$*\; y'' + y' - xy = 0 \;,\; \underline{y(0) = 2} \;,\; \underline{y'(0) = 1}$

$\{X_0 = 0\}$ معناها

$P(x) = 1 \longrightarrow P(0) = 1 \neq \infty$

$Q(x) = -x \longrightarrow Q(0) = 0 \neq \infty$ $\Big\} \longrightarrow (o.P)$

$$y = \sum a_n x^n$$

$$y' = \sum n a_n x^{n-1} \longrightarrow y'' = \sum n(n-1) a_n x^{n-2}$$

$$\sum n(n-1) a_n x^{n-2} + \sum n a_n x^{n-1} - \sum a_n x^{n+1} = 0$$

Coeff. of x^0

$2a_2 + a_1 -$ لا يوجد $= 0 \Rightarrow \boxed{a_2 = \frac{-a_1}{2}}$

Coeff. of x^1

$$6a_3 + 2a_2 - a_0 = 0 \Rightarrow a_3 = \frac{a_0 - 2a_2}{6}$$

$$\boxed{a_3 = \frac{a_1 + a_0}{6}}$$

$\rightarrow$ Coeff. of x^n

$$(n+1)(n+2)a_{n+2} + (n+1)a_{n+1} - a_{n-1} = 0$$

$$\boxed{a_{n+2} = \frac{-(n+1)\,a_{n+1} + a_{n-1}}{(n+2)(n+1)}}$$

$\underline{n=2}$

$$a_4 = \frac{-3a_3 + a_1}{12} = \frac{a_0}{24} + \frac{a_1}{24}$$

$\underline{n=3}$ $\quad a_5 = \frac{a_0}{120} - \frac{a_1}{120}$

$$\boxed{y = a_0 + a_1 x - \frac{a_1}{2}x^2 + \frac{a_1 + a_0}{6}x^3 + \left(\frac{a_0}{24} + \frac{a_1}{24}\right)x^4 + \left(\frac{a_0}{120} - \frac{a_1}{120}\right)x^5 \cdots}$$

$\boxed{y(0) = 2 = a_0}$

تعدد بدلالة x

$y' = a_1 - a_2 x \cdots$

$\boxed{y'(0) = a_1 = 1}$

نعوض في القانون.

* $(x^2+4)y'' + xy = x+2$, $x_0 = 0$

$\boxed{\text{Sol}}$

$$\left.\begin{aligned} P(x) &= 0 \longrightarrow P(0) = 0 \neq \infty \\ Q(x) &= \frac{x}{x^2+4} \longrightarrow Q(0) = 0 \neq \infty \end{aligned}\right\} \longrightarrow \text{O.P}$$

$$y = \sum a_n x^n \longrightarrow y'' = \sum n(n-1)a_n x^{n-2}$$

$$\sum n(n-1)a_n x^n + \sum 4n(n-1)a_n x^{n-2} + \sum a_n x^{n+1}$$

$$= x+2$$

Coeff. of x^0

$$0 + 8a_2 = 2 \longrightarrow \boxed{a_2 = \frac{1}{4}}$$

Coeff. of x^1

$$0 + 24a_3 + a_0 = 1 \Rightarrow \boxed{a_3 = \frac{-a_0+1}{24}}$$

Coeff of x^n

$$n(n-1)a_n + 4(n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

$$a_{n+2} = \frac{-n(n-1)a_n - a_{n-1}}{4(n+2)(n+1)}$$

$n=2$

$$a_4 = \frac{-2a_2 - a_1}{48} = \frac{\frac{-1}{2} - a_1}{48}$$

$n=3$ $\quad a_5 = \cdots$

$$y = a_0 + a_1 x + \frac{1}{4} x^2 + \frac{-a_0+1}{24} x^3 + \frac{\frac{-1}{2} - a_1}{48} x^4 \cdots$$

Solve

$$x y'' + \sin x \, y = 0 \qquad x_0 = 0$$

$$x y'' + \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right] y = 0$$

$P(x) = 0 \longrightarrow P(0) = 0 \neq \infty$

$Q(x) = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots\right) \longrightarrow Q(0) = 1 \neq \infty$

o.P

$$y = \sum a_n x^n$$

$$\therefore \sum n(n-1) a_n x^{n-1} + \sum a_n x^{n+1} - \frac{1}{3!} \sum a_n x^{n+3}$$

$$+ \frac{1}{5!} \sum a_n x^{n+5} \cdots\cdots = 0$$

Coeff. of x^0

$0 + \underset{a_{-1}}{\text{لايوجد حاجة}} + \underset{a_{-3}}{\text{لايوجد حد}} \ldots$

$\longrightarrow$ لا يوجد معامل ل x^0

Coeff. of x^1

$$2a_2 + a_0 = 0 \Longrightarrow \boxed{a_2 = \frac{-a_0}{2}}$$

Coeff. of x^n

$$(n+1)n\,a_{n+1} + a_{n-1} - \frac{1}{3!}a_{n-3} + \frac{1}{5!}a_{n-5} \cdots = 0$$

$$\boxed{a_{n+1} = \frac{-a_{n-1} + \frac{1}{3!}a_{n-3} - \frac{1}{5!}a_{n-5} \cdots}{n(n+1)}}$$

$n = 2$

$$\boxed{a_3 = \frac{-a_1}{6}}$$

$\boxed{n = 3}$

$$a_4 = \frac{-a_2 - \frac{1}{3!}a_0}{12} = \frac{\frac{a_0}{2} - \frac{a_0}{6}}{12}$$

$$y = a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 \cdots$$

$\boxed{2}$ $S.P \longrightarrow P(0) = \infty$ or $Q(0) = \infty$

a) I.S.P $\longrightarrow$ $\left. x P(x) \right|_{x=0} = \infty$ or $\left. x^2 Q(x) \right|_{x=0} = \infty$ } ليس لها حل

b) R.S.P $\left. x \cdot P(x) \right|_{x=0} \neq \infty$

and $\left. x^2 \cdot Q(x) \right|_{x=0} \neq \infty$

$$y = \sum a_n x^{n+\lambda} \qquad y_{G.S} = c_1 y_1 + c_2 y_2$$

$\lambda_1 - \lambda_2 \neq$ عدد صحيح

$\lambda_1 = \lambda_2 = \lambda$

$\lambda_1 - \lambda_2 =$ عدد صحيح

← قوانين هذا ~~[illegible]~~ الجزء في المحاضرة

$$* \quad x^2 y'' + x y' + \left(x^2 - \frac{4}{9}\right) y = 0$$

$$P(x) = \frac{1}{x} \longrightarrow P(0) = \infty$$

$$Q(x) = 1 - \frac{4}{9x^2} \longrightarrow Q(0) = \infty$$

} $\longrightarrow S.P$

$$\left.\begin{array}{l} x\,P(x)\big|_{x=0} = 1 \neq \infty \\ x^2\,Q(x)\big|_{x=0} = x^2 - \frac{4}{9} = \frac{-4}{9} \neq \infty \end{array}\right\} R.S.P$$

$$y = \sum a_n\, x^{n+\lambda}$$

$$y' = \sum (n+\lambda)\, a_n\, x^{n+\lambda-1}$$

$$y'' = \sum (n+\lambda)(n+\lambda-1)\, a_n\, x^{n+\lambda-2}$$

$$\boxed{\begin{array}{l} \sum (n+\lambda)(n+\lambda-1)\, a_n\, x^{n+\lambda} + \sum (n+\lambda)\, a_n\, x^{n+\lambda} \\ + \sum a_n\, x^{n+\lambda+2} - \frac{4}{9} \sum a_n\, x^{n+\lambda} = 0 \end{array}}$$

@ $\underline{n = 0}$ smallest $\rightarrow x^{\lambda}$

$\therefore$ Coeff x^{λ}

$$\lambda(\lambda-1)\, a_0 + \lambda a_0 - \frac{4}{9} a_0 = 0$$

$$a_0\left(\lambda^2 - \lambda + \lambda - \frac{4}{9}\right) = 0$$

$$a_0\left(\lambda^2 - \frac{4}{9}\right) = 0$$

$$\lambda^2 = \frac{4}{9} \longrightarrow \lambda_1 = \frac{2}{3} \quad , \quad \lambda_2 = \frac{-2}{3}$$

$$\lambda_1 - \lambda_2 = \frac{4}{3} \neq$$ عدد صحيح

$\longrightarrow$ Coeff. of $x^{\lambda+1}$

$$(\lambda+1)\lambda a_1 + (\lambda+1) a_1 - \frac{4}{9} a_1 = 0$$

$$a_1 \left[\lambda^2 + \lambda + \lambda + 1 - \frac{4}{9}\right] = 0 \qquad \boxed{a_1 = 0}$$

Coeff. of $x^{\lambda+n}$

$$(n+\lambda)(n+\lambda-1) a_n + (n+\lambda) a_n + a_{n-2} - \frac{4}{9} a_n = 0$$

$$a_n = \frac{-a_{n-2}}{(n+\lambda)(n+\lambda-1) + (n+\lambda) - \frac{4}{9}}$$

$n = 1$

$$\cancel{a_n = \frac{-a_0}{(2+\lambda) - \frac{4}{9}}} \qquad a_2 = \text{[illegible]}$$

$$y_1 = x^{\frac{2}{3}} \left[a_0 + \frac{-a_0}{\left(2+\frac{2}{3}\right)^2 - \frac{4}{9}} x^2 \cdots\cdots\right]$$

$$y_2 = x^{\frac{-2}{3}} \left[a_0 + \frac{-a_0}{\left(2-\frac{2}{3}\right)^2 - \frac{4}{9}} x^2 \cdots\right]$$

$$y_{G.S} = C_1 y_1 + C_2 y_2$$

$$* \quad x^2 y'' - x y' + (1+x) y = 0$$

$$P(x) = \frac{-1}{x} \longrightarrow P(0) = \infty \quad \Rightarrow x\, P(x)\big|_{x=0} = -1$$

S.P

$$Q(x) = \frac{1}{x^2} + \frac{1}{x} \longrightarrow Q(0) = \infty \quad \Rightarrow x^2 Q(x)\big|_{x=0} = 1 \neq \infty$$

R.S.P

$$y = \sum a_n x^{n+\lambda}$$

$$y' = \sum (n+\lambda) a_n x^{n+\lambda-1}$$

$$y'' = \sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2}$$

$$\boxed{\sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda} - \sum (n+\lambda) a_n x^{n+\lambda} + \sum a_n x^{n+\lambda} + \sum a_n x^{n+\lambda+1} = 0}$$

$n = 0 \longrightarrow x^{\lambda}$ أقل أس هو

Coeff. of x^{λ}

$$\lambda(\lambda-1) a_0 - \lambda a_0 + a_0 = 0$$

$$a_0 (\lambda^2 - 2\lambda + 1) = 0$$

$$\boxed{\lambda = \lambda_1 = \lambda_2 = 1}$$

$*$ Coeff. of $x^{\lambda+1}$

$$(\lambda+1)\lambda a_1 - (\lambda+1) a_1 + a_1 + a_0 = 0$$

$$\boxed{a_1 = \frac{-a_0}{\lambda^2}}$$

$\rightarrow$ Coeff. of $x^{\lambda+n}$

$$(n+\lambda)(n+\lambda-1) a_n - (n+\lambda) a_n + a_n + a_{n-1} = 0$$

$$a_n = \frac{-a_{n-1}}{(n+\lambda)(n+\lambda-1) - (n+\lambda) + 1}$$

$$a_n = \frac{-a_{n+1}}{(n+\lambda)(n+\lambda-2) + 1}$$

$n=2$ $\quad a_2 = \dfrac{-a_1}{(\lambda+2)(\lambda)+1} = \dfrac{-a_1}{(\lambda+1)^2}$

$$y_1 = x\left[a_0 - a_0 x + \frac{a_0}{4} x^2 \ldots\ldots\right]$$

$$y(x,\lambda) = x^{\lambda}\left[a_0 - a_0 \lambda^{-2} x - a_0 (\lambda^2+\lambda)^{-2} x^2 \ldots\right]$$

$$x^2 y'' + (x^2 - 2x)\,y' + 2y = 0$$

$$P(x) = \frac{x^2 - 2x}{x^2} = 1 - \frac{2}{x} \rightarrow P(0) = \infty \rightarrow x\,P(x) = x - 2 = -2 \neq \infty$$

$$Q(x) = \frac{2}{x^2} \longrightarrow Q(0) = \infty \longrightarrow x^2 Q(x) = 2 \neq \infty$$

$\}\rightarrow$ R.S.P

$$y = \sum a_n x^{n+\lambda}$$

$$y' = \sum (n+\lambda)\, a_n x^{n+\lambda-1}$$

$$y'' = \sum (n+\lambda)(n+\lambda-1)\, a_n x^{n+\lambda-2}$$

$$\boxed{\sum (n+\lambda)(n+\lambda-1)\, a_n x^{n+\lambda} + \sum (n+\lambda)\, a_n x^{n+\lambda+1} - 2 \sum (n+\lambda)\, a_n x^{n+\lambda} + 2 \sum a_n x^{n+\lambda} = 0}$$

Coeff. of x^{λ}

$$\lambda(\lambda-1)\, a_0 - 2\lambda\, a_0 + 2a_0 = 0$$

$$a_0(\lambda^2 - \lambda - 2\lambda + 2) = 0$$

$$a_0(\lambda^2 - 3\lambda + 2) = 0$$

$$a_0\big((\lambda-1)(\lambda-2)\big) = 0 \qquad \lambda = 1 \;,\; \lambda = 2$$

$-$ = [illegible] 1

Coeff. of $x^{\lambda+1}$

$$\lambda(\lambda+1)\, a_1 + \lambda\, a_0 - 2(\lambda+1)\, a_1 + 2a_1 = 0$$

$$a_1 = \frac{-\lambda a_0}{\lambda^2 + \underbrace{\lambda - 2\lambda}_{-\lambda} \cancel{-2} \cancel{+2}} = \frac{-a_0}{\lambda - 1}$$

Coeff. of $x^{n+\lambda}$

$$(n+\lambda)(n+\lambda-1)a_n + (n+\lambda-1)a_{n-1} - 2(n+\lambda)a_n + 2a_n = 0$$

$$a_n = \frac{-(n+\lambda-1)a_{n-1}}{(n+\lambda)(n+\lambda-1) + (n+\lambda-1) - 2(n+\lambda) + 2}$$

$$a_n = \frac{-(n+\lambda-1)a_{n-1}}{(n+\lambda-1)(n+\lambda+1) - 2(n+\lambda-1)} = \frac{-a_{n-1}}{n+\lambda+1-2}$$

$$a_n = \frac{-a_{n-1}}{n+\lambda-1}$$

$n=2$ $\quad a_2 = \frac{-a_1}{\lambda+1} = \boxed{\frac{a_0}{\lambda^2-1}}$

$\lambda=2$ $\quad y_1 = x^2\left[a_0 - \frac{a_0}{\lambda-1}x + \frac{a_0}{\lambda^2-1}x^2 + \cdots\right]$

$\lambda=1 \longrightarrow a_2 = \infty$

$$y(x,\lambda) = x^\lambda\left[a_0 - a_0(\lambda-1)^{-1}x + a_0(\lambda^2-1)^{-1}x^2 + \cdots\right]$$

$$(\lambda-1)\,y(x,\lambda) = x^\lambda\left[a_0(\lambda-1) - a_0x + a_0(\lambda+1)x^2 + \cdots\right]$$

$$\frac{\partial}{\partial\lambda} = x^\lambda\left[a_0 + a_0x^2 + \cdots\right]$$
$$+\, x^\lambda\ln(x)\left[a_0(\lambda-1) - a_0x + a_0(\lambda+1)x^2 + \cdots\right.$$

$$y_2 = x\left[a_0 + a_0x^2 + \cdots\right] + x\ln(x)\left[-a_0x + 2a_0x^2 + \cdots\right]$$

$\boxed{y_{(x)} = c_1y_1 + c_2y_2}$ #